If what you need is ….	then the rule to use is…	and what else you need is…
The P in PvQ	vE	~Q
The Q in PvQ	vE	~P
The P in P&Q	&E	nothing
The Q in P&Q	&E	nothing
The P→Q in P↔Q	↔E	nothing
The Q in P→Q	→E	P
P in P→Q	(You just can't get there from here)	Another strategy

If what you need to build is …	Then the rule to use is …	and what you need is …	… and
P&Q	&I	P	Q
PvQ	vI	either P or Q	(nothing)
P→Q	→I	…to assume P	…and deduce Q
~P	RAA	…to assume P	…and deduce both X and ~X (for any wff X)
P↔Q	↔I	P→Q	O→P

Symbolic logic

Strategies for less simple proofs

Chapman University. PHIL300. Lecture 11. 3/8/2022.
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The road to in-class exam 2

Today: section 1.5.

- Less simple proofs.

Lectures 12 (3/10): section 1.5.

- Derived rules.

Lecture 13 (3/15): section 1.6.

- Theorems.

Lecture 14 (3/17): revision.

Lecture 15 (3/29): in-class exam 2.

- Covers sections 1.4,1.5 and 1.6.

Truth tables versus proofs

- Truth tables
  - By using the five truth tables (for the five connectives), we can prove the validity of many arguments.
  - But they don't give us much insight into actual reasoning, and especially, how to go about building our own logical arguments.

- Proofs
  - Embedded within each truth table is an *introduction* and an *elimination* rule (~ is the exception).
  - These represent *the foundations of logical reasoning* and are used (and misused) in all arguments we see in everyday life, philosophy, science, and mathematics.

Examples: arrow and wedge elimination

- Arrow-elim
  - Given a conditional sentence (at line *m*) and another sentence that is its antecedent (at line *n*), conclude the consequent of the conditional.

    (*m*): If I'm going to get an A in the next exam, then I *must* do lots of proof exercises.

    (*n*): I *am* going to get an A in the next exam.

    (*conclude*): I *must* do lots of proof exercises.

P	→	Q
T	T	T
T	F	F
F	T	T
F	T	F

- Wedge-elim
  - Given a sentence (at line *m*) that is a disjunction and another sentence (at line *n*) that is a denial of one of its disjuncts, conclude the other disjunct.

    (*m*): I'm going to do lots of proof exercises or I'm going to get a mediocre grade in the next exam.

    (*n*): I'm *not* going to getting a mediocre grade in the next exam.

    (*conclude*): I'm going to do lots of proof exercises.

P	v	Q
T	T	T
T	T	F
F	T	T
F	F	F

Arrow introduction

- Arrow-intro
  - Given a sentence (at line *n*) conclude a conditional having it as the consequent and whose antecedent appears in the proof as an assumption (at line *m*).

(*premise*): If all choices are determined, then no choices are free.

(*premise*): If no choices are free, then no one is responsible for what they do.

(*n*): [*Assume for argument's sake that*] all choices are determined.

(*m*): No one is responsible for what they do.

(*conclude*): If all choices are determined, then no one is responsible for what they do.

P	→	Q
T	T	T
T	F	F
F	T	T
F	T	F

Reductio ad absurdum (RAA)

- RAA
  - Given both a sentence and its denial (at lines *m* and *n*), conclude the denial of any assumption appearing in the proof (at line *k*).
    - Here, the intuition is that *if an assumption is responsible for a contradiction* (a sentence and its denial both being true), *that assumption must be false.*
    - So, if you want to prove something (anything), assume its denial and show that the assumption leads to a contradiction.

Example: prove that there's **no** largest number.

(*premise l*): If there is a largest number, N, then there is no such number as N+1.

(*premise m*): There is such a number as N+1.

(*k*): [*Assume for argument's sake that*] There **is** a largest number, N.

(*n*): There is no such number as N+1 (*from premise l and assumption k*)

(*conclude*): There is no largest number (*since (k) led to the contradiction in m and n*).

Strategies for less simple proofs

- First ask: what am I trying to prove?
  - A proof is a trip to a given destination (the conclusion) from a given starting point (the premises).
  - Your task is to discover *a* route, but not *the* route, since there are often many different ones.
- A general strategy for getting to where you want to go from where you already are is to *work backwards from your destination*.
  - Start with where you want to end up (the conclusion), and figure out how you could get there.
  - Keep working backwards until you find that what you need is something you know how to get from the premises.

Know how to identify main connectives

- In sentential logic there are exactly *six* kinds of well-formed-formula (wff). Every wff is:
    1. Atomic (a letter all by itself), or
    2. a Negation (main connective: ~), or
    3. a Conjunction (main connective: &), or
    4. a Disjunction (main connective: v), or
    5. a Conditional (main connective: →), or
    6. a Biconditional (main connective: ↔).
- The main connective of the wff you're working with determines both the strategies for proving it and the strategies for using it in proving other things.

The direct approach

1. Do I already have the wff that I need to prove, as a constituent part of something I already have?
2. If I do, how do I take that thing apart and get out the wff that I need?
   - The *Elimination* rules are useful here.
3. If I don't, then how do I build what I need?
   - The *Introduction* rules are useful here.

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The indirect approach

- Sometimes, the direct approach doesn't work.
  - In that case, you can always resort to assuming a denial of what you want to get, then try to get a contradiction so that you can use RAA.

P v P ⊢ P

1	(1)	P v P	A

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P v P ⊢ P

1	(1)	P v P	A
2	(2)	~P	A
1,2	(3)	P	1,2vE
1	(4)	P	2,3 RAA(2)

Double turnstile (p28)

Comment. If a sequent has just one sentence on each side of a turnstile, a reversed turnstile may be inserted ($\dashv$) to represent the argument from the sentence on the right to the sentence on the left.

Example. $P \dashv\vdash P \vee P$

Comment. This example corresponds to two sequents: $P \vdash P \vee P$ and $P \vee P \vdash P$. You may read the example as saying ‘P therefore P or P, and P or P therefore P’. When proving $\phi \dashv\vdash \psi$, one must give two proofs: one for $\phi \vdash \psi$ and one for $\psi \vdash \phi$.

Exercise 1.5.1 Give proofs for the following sequents, using the primitive rules of proof.

S11*	$P \dashv\vdash {\sim}{\sim}P$	**D**ouble **N**egation
S12*	$P \to Q, {\sim}Q \vdash {\sim}P$	**M**odus **T**ollendo **T**ollens
S13	$P \to {\sim}Q, Q \vdash {\sim}P$	MTT
S14*	${\sim}P \to Q, {\sim}Q \vdash P$	MTT
S15	${\sim}P \to {\sim}Q, Q \vdash P$	MTT
S16*	$P \to Q, Q \to R \vdash P \to R$	**H**ypothetical **S**yllogism
S17*	$P \vdash Q \to P$	**T**rue **C**onsequent
S18*	${\sim}P \vdash P \to Q$	**F**alse **A**ntecedent
S19	$P \vdash {\sim}P \to Q$	**FA**
S20	$P \to Q, P \to {\sim}Q \vdash {\sim}P$	**I**mpossible **A**ntecedent
S21*	${\sim}P \vee Q \dashv\vdash P \to Q$	Wedge-Arrow ($\vee\to$)
S22	$P \vee Q \dashv\vdash {\sim}P \to Q$	$\vee\to$
S23	$P \vee Q \dashv\vdash {\sim}Q \to P$	$\vee\to$
S24	$P \vee {\sim}Q \dashv\vdash Q \to P$	$\vee\to$
S25	$P \vee Q, P \to R, Q \to R \vdash R$	**Simp**le **Dil**emma
S26*	$P \vee Q, P \to R, Q \to S \vdash R \vee S$	**Comp**lex **Dil**emma
S27	$P \to Q, {\sim}P \to Q \vdash Q$	**Spec**ial **Dil**emma
S28*	${\sim}(P \vee Q) \dashv\vdash {\sim}P \,\&\, {\sim}Q$	**DeM**organ's Law
S29	${\sim}(P \,\&\, Q) \dashv\vdash {\sim}P \vee {\sim}Q$	DM